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# Strang-Fix Theory for Approximation Order in Weighted $L^p$ -spaces

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We consider the Strang-Fix theory for approximation order in the weighted  $L^p$ -spaces. Let  $\varphi$  be an element of  $C_c(\mathbb{R}^n)$ . For a sequence  $c$  on  $\mathbb{Z}^n$ , the semi-discrete convolution product  $\varphi *' c$  is the function defined by

$$\varphi *' c = \sum_{\nu \in \mathbb{Z}^n} \varphi(\cdot - \nu) c(\nu).$$

The collection  $\Phi = \{\varphi_1, \dots, \varphi_N\}$  of  $C_c(\mathbb{R}^n)$  is said to satisfy the Strang-Fix condition of order  $k$  if there exist finitely supported sequences  $b_j$  ( $j = 1, \dots, N$ ) such that the function  $\varphi = \sum_{j=1}^N \varphi_j *' b_j$  satisfies

$$\hat{\varphi}(0) \neq 0$$

and

$$(\partial^\alpha \hat{\varphi})(2\pi\nu) = 0 \quad (|\alpha| < k, \quad \nu \in \mathbb{Z}^n \setminus \{0\}),$$

where  $\hat{\varphi}$  denotes the Fourier transform of  $\varphi$ . For a positive integer  $k$ ,  $L_k^p(\mathbb{R}^n)$  denotes the Sobolev space. For  $f \in L_k^p(\mathbb{R}^n)$ , we define semi-norms by

$$|f|_{k,p} = \sum_{|\alpha|=k} \|\partial^\alpha f\|_{L^p(\mathbb{R}^n)}.$$

For  $h > 0$ ,  $\sigma_h$  is the scaling operator defined by

$$\sigma_h f(x) = f(hx) \quad (x \in \mathbb{R}^n).$$

We say that  $\Phi = \{\varphi_1, \dots, \varphi_N\}$  provides local  $L^p$ -approximation of order  $k$  if there exist constants  $C$  and  $r$  such that for each  $f \in L_k^p(\mathbb{R}^n)$  there exist sequences  $c_j^h$  ( $h > 0$ ,  $j = 1, \dots, N$ ) so that

$$(i) \quad \|f - \sigma_{1/h}(\sum_{j=1}^N \varphi_j *' c_j^h)\|_{L^p(\mathbb{R}^n)} \leq Ch^k |f|_{k,p},$$

(ii)  $c_j^h(\nu) = 0$  ( $j = 1, \dots, N$ ) whenever  $\text{dist}(h\nu, \text{supp } f) > r$ .

Boor and Jia [1] proved that  $\Phi$  satisfies the Strang-Fix condition of order  $k$  if and only if  $\Phi$  provides local  $L^p$ -approximation of order  $k$ .

We give the definition of  $A_p$  in  $\mathbb{R}^n$ . A weight  $w \geq 0$  is said to belong to  $A_p$  for  $1 < p < \infty$  if

$$A_p(w) = \sup_Q \left( \frac{1}{|Q|} \int_Q w(x) dx \right) \left( \frac{1}{|Q|} \int_Q w(x)^{1-p'} dx \right)^{p-1} < \infty,$$

where  $Q$  is a cube in  $\mathbb{R}^n$  and  $p'$  is a conjugate exponent of  $p$ .  $A_p(w)$  is called the  $A_p$ -constant of  $w$ . The class  $A_1$  is defined by

$$A_1(w) = \sup_Q \left( \frac{1}{|Q|} \int_Q w(x) dx \right) \|w^{-1}\|_{L^\infty(Q, dx)} < \infty,$$

where  $\|w^{-1}\|_{L^\infty(Q, dx)} = \text{ess sup}_{x \in Q} w(x)^{-1}$ . The class  $A_\infty$  is the union of the classes of  $A_p$ ,  $1 \leq p < \infty$ . These classes were introduced by Muckenhoupt in [3]. Let  $1 \leq p \leq \infty$  and  $w \in A_p$ . Then the weighted  $L^p$ -space  $L^p(w)$  consists of all measurable functions on  $\mathbb{R}^n$  such that

$$\|f\|_{L^p(w)} = \left( \int_{\mathbb{R}^n} |f(x)|^p w(x) dx \right)^{1/p} < \infty,$$

with the usual modifications when  $p = \infty$ . We define the weighted Sobolev spaces  $L_k^p(w)$ , where  $1 \leq p \leq \infty$ ,  $w$  is an  $A_p$ -weight and  $k$  is a positive integer. A function  $f$  belongs to  $L_k^p(w)$  if  $f \in L^p(w)$  and the partial derivatives  $\partial^\alpha f$ , taken in the sense of distributions, belong to  $L^p(w)$ , whenever  $0 \leq |\alpha| \leq k$ . The norm in  $L_k^p(w)$  is given by

$$\|f\| = \sum_{|\alpha| \leq k} \|\partial^\alpha f\|_{L^p(w)}.$$

In the weighted case, we use the following notation

$$|f|_{k,p,w} = \sum_{|\alpha|=k} \|\partial^\alpha f\|_{L^p(w)}.$$

and say that  $\Phi = \{\varphi_1, \dots, \varphi_N\}$  provides local  $L^p(w)$ -approximation of order  $k$  if there exist constants  $C$  and  $r$  such that for each  $f \in L_k^p(w)$  there exist sequences  $c_j^h$  ( $h > 0$ ,  $j = 1, \dots, N$ ) so that (ii) and the following condition (iii) are satisfied

$$(iii) \quad \|f - \sigma_{1/h} \left( \sum_{j=1}^N \varphi_j * c_j^h \right)\|_{L^p(w)} \leq Ch^k |f|_{k,p,w}.$$

Based on [2], using boundedness of the Hardy-Littlewood maximal operator on  $L^p(w)$ , we prove the following theorem.

**Theorem.** *Let  $\Phi = \{\varphi_1, \dots, \varphi_N\}$  be a finite collection of  $C_c(\mathbb{R}^n)$ . Then the following statements are equivalent.*

- (i)'  $\Phi$  satisfies the Strang-Fix condition of order  $k$ .
- (ii)' For all  $p \in [1, \infty]$  and  $w \in A_p$ ,  $\Phi$  provides local  $L^p(w)$ -approximation of order  $k$ .
- (iii)' For some  $p \in [1, \infty]$  and  $w \in A_p$ ,  $\Phi$  provides local  $L^p(w)$ -approximation of order  $k$ .

Lastly we introduce the main lemma to prove the above theorem.

**Lemma.** *Let  $1 \leq p < \infty$  and  $w \in A_p$ . Suppose that  $\varphi$  is a function on  $\mathbb{R}^n$  which is non-negative, radial, decreasing and integrable. Then there exists a constant  $C$  such that*

$$\int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} |f(x + \alpha y)| \varphi(y) dy \right)^p w(x) dx \leq C \int_{\mathbb{R}^n} |f(x)|^p w(x) dx$$

for all  $f \in L^p(w)$  and  $\alpha \in \mathbb{R}$ .

N. Tomita proved the above lemma when  $1 < p < \infty$ , using Calderón-Zygmund Operator. Then Professor E. Nakai provided the simple proof when  $1 < p < \infty$ , using Hardy-Littlewood maximal operator. Then Professor K. Yabuta proved the case  $p = 1$ .

## References

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